ME404 Response function method

There are essentially two methods that can be employed to represent the dynamic interactions that occur within an energy system: *response function* and *numerical*. Each method provides a solution to the differential equations that govern the flow of heat in solids, heat transfer at surface layers and heat exchange between connected fluid volumes. The response function approach is usually applied to differential problems of low order with time-invariant parameters whereas the numerical method is also suited to time varying problems of high order. This handout describes the response function method.

The response function method is often used to estimate the internal air temperature prevailing in an unconditioned building or the heating/cooling requirements to maintain a specified temperature. A detailed consideration of the method is outwith the scope of this course; what follows is a high level description.

Essentially, the method provides an analytical solution to the partial differential equations representing an energy system via a Laplace transformation as follows.

1. The given equations in the time domain, f(t), are transformed to a subsidiary equation in an imaginary space, f(p).
2. The subsidiary equations are solved by purely algebraic manipulations.
3. An inverse transformation is applied to the solution to obtain the solution in the time domain of the initial problem.

The interesting feature of the method is that in many cases ordinary differential equations are transformed into purely algebraic equations and partial differential equations are transformed to ordinary differential equations – thus facilitating an analytical solution. Transforms and inverse transforms can sometimes be obtained from Laplace transform tables such as shown in table 1; otherwise they are evaluated mathematically.

Table 1: Some common Laplace transform pairs.

	f(t)	f(p)
Unit impulse	$\delta(t)$	1
Unit step	$H(t)$	$1/p$
Unit ramp	t	$1/p^2$
	t^n	$n!/p^{n+1}$; n +ve integer
Delayed unit impulse	$\delta(t-\Delta)$	$e^{-p\Delta}$
Delayed unit step	$H(t-\Delta)$	$e^{-p\Delta}/p$
	e^{-at}	$1/(p+a)$
	$e^{-a(t-\Delta)}\,H(t-\Delta)$	$e^{-p\Delta}/(p+a)$
	te^{-at}	$1/(p+a)^2$
	$t^n e^{-at}$	$n!/(p+a)^{n+1}$
	$\sin bt$	$b/(p^2+b^2)$
	$\cos bt$	$p/(p^2+b^2)$

A Laplace transformation may be undertaken in the time or frequency domain: the former concerned with the response of a system to time-varying boundary temperatures or fluxes; the latter with the response to periodic excitations of differing frequencies. Because the frequency domain approach in its simplest form is amenable to manual application it is considered further here. In the UK, this approach is known as the Admittance method.

The Admittance method

The fundamental assumption underlying the response function method in the frequency domain is that weather time-series can be represented by a series of periodic cycles. In this way the weather's influence is represented by a steady-state term accompanied by a number of sine wave harmonics with, in general, increasing frequency and reducing amplitude. This division of weather time-series into component sinusoidal variations about a mean condition may be achieved via a Fourier series representation. Each harmonic can then be processed separately and modified by thermal response factors as determined in advance from the application of the Laplace transform method to representative cases. The overall system response is then obtained by summing the individual effects of the separate harmonics with respect to the mean condition. For convenience, the frequency of the fundamental harmonic is often set at 24 hours with the remaining harmonics having diminishing periods such as 12, 6, 3, 1.5 hours etc.

The Admittance method corresponds to the case where only the mean condition and the 24 hour harmonic are employed. Three principal response factors are then used: decrement response, surface response and admittance response, each possessing a corresponding phase angle that determines the response time of the process the factor addresses.

The *decrement response factor* is defined as the ratio of the cyclic flux transmission to the steady state flux transmission and is applied to fluctuations (about the mean) in external temperature or flux impinging on exposed constructions. This gives the related fluctuation within the building at some later point in time depending on the decrement factor time lag. Figure 1 shows a wall exposed to a sinusoidal external air temperature or solar radiation time-series. The corresponding cyclic heat flux at the inside surface (x=0) is also shown after time-series modification by the decrement factor of the intermediate elements and application of the appropriate time shift.

The *surface response factor* defines the portion of the heat flux at an internal surface which is re-admitted to the internal environmental point when temperatures are held constant. The factor is applied to cyclic energy inputs at an internal surface to give the corresponding cyclic energy variations at the environmental point; the concept is analogous to time-shifted reflections as illustrated in figure 2. Typical applications include the modification of the transmitted component of solar radiation through windows and the radiant component of casual gains both of which strike internal surfaces.

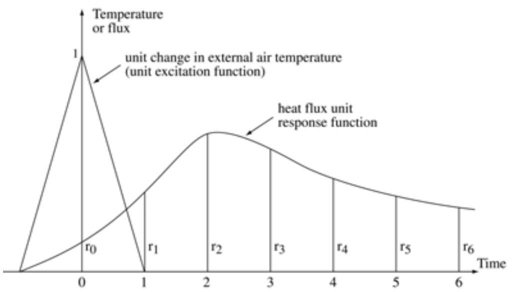

Figure 1: Decrement factor and time lag.

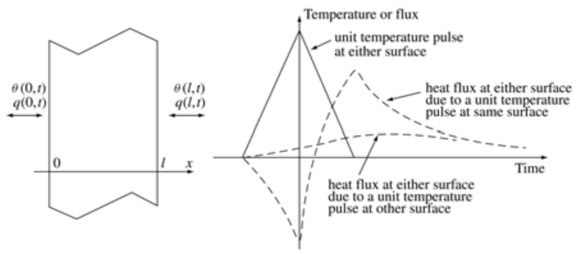

Figure 2: Surface factor and time lag.

The admittance response factor is defined as the amount of energy entering a surface for each degree of temperature swing at the environmental point. It is used to represent enclosure response and give the equivalent swing in temperature about the mean value due to a cyclic heat load on the enclosure.

By adding the cyclic contribution from the 24 hour harmonic to the mean condition, performance predictions of internal temperature, heating/cooling requirements and the effects of variable ventilation and intermittent plant operation may be made. The procedure for internal temperature estimation at some point in time is as follows.

First, the mean internal environmental temperature is determined i from

$$\theta'_{ei} = \theta'_{ao} + Q'_t \Big/ \left(\sum_{i=1}^{c} A_i U_i + C'_v \right)$$

where θ'_{ei} is the daily mean internal environmental temperature (°C), θ'_{ao} the mean external air temperature (°C), Q'_t the mean total heat flux from all sources (W), C'_v the mean ventilation conductance (W/°C), $\Sigma A_i U_i$ the sum of the product of areas and overall thermal transmittance values (W/°C) and c is the number of constructions.

The mean total heat flux is given by

$$Q'_t = Q'_{fs} + Q'_s + Q'_c$$

where Q'_{fs} is the mean solar gain through opaque surfaces, Q'_s the mean solar gain through transparent surfaces, and Q'_c the mean gain from casual sources (all measured in W). These terms are now considered in turn.

$$Q'_{fs} = \sum_{i=1}^{N} \left[A_i U_i R_{oi} \left(\alpha_i I'_{soi} - \varepsilon_i I_{oi} \right) \right]$$

where N is the total number of exposed opaque surfaces, α_i the surface shortwave absorptivity, $\varepsilon_i I_{oi}$ the longwave radiation exchange with the surroundings (W/m^2), R_{oi} the combined surface resistance (m^2°C/W) and I'_{soi} the mean solar flux incident on the opaque surface:

$$\sum_{t=1}^{M} I_{soi}(t) \Big/ m$$

where m=24 for the daily mean and $I_{soi}(t)$ the instantaneous solar flux impinging on opaque surface i (W/^{2}).

The mean solar gain through transparent surfaces is found from

$$Q'_s = \sum_{t=1}^{M} Q_s(t) \Big/ m \; .$$

To determine the portion of the incident solar flux that penetrates windows, predetermined solar gain factors are employed:

$$Q_s(t) = \sum_{i=1}^{L} A_i S_i(t) I_{soi}(t)$$

where L is the total number of transparent surfaces and $S_i(t)$ the solar gain factor at time t.

$$Q_c(t) = \sum_{j=1}^{K} Q_{cj}(t)$$

where K is the number of casual sources and Q_{cj} the magnitude of any source (W).

<u>Second</u>, the contribution of the 24 hour harmonic to the swing in internal environmental temperature about the mean value is then computed from

$$\theta''_{ei}(t) = \frac{Q''(t - \Phi_a)}{\sum A.a + C_v}$$

where $\theta''_{ei}(t)$ is the fluctuation in internal environmental temperature about the mean at time t (°C), $Q''_t(t-\Phi_a)$ the total fluctuating gain at the environmental point at time t-Φ_a (W); Φ_a the time lag associated with the admittance factor (s), and ΣA.a the sum of the produce of area and admittance for all internal surfaces (W/°C).

The total fluctuating gain at the environmental point is given by

$$Q''_t(t) = Q''_{fs}(t) + Q''_s(t) + Q''_c(t) + Q''fc(t) + Q''gc(t) + Q''v(t)$$

where $Q''_{fs}(t)$ is the opaque surface solar gain fluctuation at time t, $Q''_s(t)$ the transparent surface solar gain fluctuation, $Q''_c(t)$$ the casual gain fluctuation, $Q''_{fc}(t)$ the opaque surface conduction gain fluctuation, $Q''_{gc}(t)$ the transparent surface conduction gain fluctuation, and $Q''_v(t)$ the ventilation or infiltration fluctuation (all measured in W). Each load fluctuation is now considered in turn.

$$Q''_{fs} = \sum_{i=1}^{N} \left[A_i U_i R_{oi} d_i \alpha_i I''_{so} (t - \Phi_d) \right]$$

where d_i is the decrement factor for the layers behind surface i, Φ_d the associated time lag, $I''_{so}(t-\Phi_d)$ the fluctuation about the mean of the solar intensity incident on opaque surfaces at some time (t- Φ_d), measured in W/m^2 and equal to $I_{so}(t-\Phi_d) - I'_{so}$.

$$Q''_s(t) = Q''_{s1}(t - \Phi_s) + Q''_{s2}(t)$$

where $Q''_{s1}(t- \Phi_s)$ is the fluctuation due to the directly transmitted (time lagged) component of solar radiation through transparent surfaces and $Q''_{s2}(t)$ the fluctuation due to the absorbed component of the incident solar radiation which is retransmitted (with no time lag) inward to the environmental point, both measured in W. The admittance method uses alternating solar gain factors to determine the fluctuation in energy at the environmental point due to the fluctuation in solar gain through transparent surfaces:

$$Q''_s(t) = \sum_{i=1}^{L} A_i S''_i (t - \Phi_s) I''_{si} (t - \Phi_s)$$

where $S''_i(t-\Phi_s)$ is the alternating solar gain factor which includes the effect of the surface response factor.

$$Q''_c(t) = Q_c(t) - Q'_c$$

where $Q''_c(t)$ is the total instantaneous casual load (W).

$$Q''_{fc}(t) = \sum_{i=1}^{o} A_i U_i d_i \theta''_{ao} (t - \Phi_d)$$

where $\theta_{ao}(t)$ is the fluctuation in outside air temperature (°C) and o the number of opaque constructions.

$$Q''_{gc}(t) = \sum_{i=1}^{T} A_i U_i \theta''_{ao}(t)$$

where T is the number of transparent constructions. It is usual practice to assume that window conduction processes undergo negligible time delay.

$$Q''_v(t) = C_v(t) \theta''_{ao}(t)$$

where, again, it is usual to assume a zero time lag.

Third, final peak temperature is obtained as the summation of the mean and fluctuating temperatures.

Admittance method: worked example

Using the data sources given in the Appendix, this section demonstrates the use of the Admittance method as described above to estimate the internal environmental temperature likely to occur at 15h00 during a warm, sunny day in August in a south facing office as described by the following data.

Latitude: 51.7°N.
Internal dimensions: 6m x 5m x 3m high.
External wall: 5m x 3m, dark external finish.
Window: 3m x 2m, not shaded, open during day, closed at night.
Occupancy: 5 persons for 8 hours at 85W per person.

Lighting: 25 W/m^2 of floor area, ON 08h00-18h00.
Construction details:

Element	U-Value ($W/m^2.K$)	Admittance ($W/m^2.K$)	Decrement (-)	Time Lag (h)
External wall: 220mm brickwork, 25mm cavity, 25mm insulation, 10mm plasterboard	0.59	0.91	0.3	8
Window: double glazed, 12mm air gap, normal exposure (ignore frame)	2.9	2.9		
Internal walls: 220mm brickwork, 13mm light plaster	1.9	3.6		
Floor: 25mm wood block, 50mm screed, 150mm cast concrete	1.5	2.9		
Ceiling: as floor but reversed	1.5	6.0		

Mean solar heat gain

$$Q_s' = S' I_t' A_g = 0.64 \; x 175 \; x (3 \; x \; 2) = 672 \; W$$

where A_g is the sunlit area of glazing, I_t' the mean total solar irradiance (from Table A8.1 in Appendix) and S' the mean solar gain factor (see Table A8.2).

Mean casual gain

$$Q_c' = \frac{g_{c1} t_1 + g_{c2} t_2 + \ldots}{24} = \frac{(5 \; x \; 85 \; x \; 8) + (6 \; x \; 5 \; x \; 25 \; x \; 10)}{24} = 454 \; W$$

where g_{c1}, g_{c2} etc. are the instantaneous casual gains and t_1, t_2, etc. the durations of g_{c1}, $g_{c2, \ldots}$.

Total mean heat gain

$$Q_t' = 672 + 454 = 1126 \; W$$

Mean internal environmental temperature

$$Q_t' = \left(\sum A_g U_g + C_v\right)\left(\theta_{ei}' - \theta_{ao}'\right) + \sum A_f U_f \left(\theta_{ei}' - \theta_{eo}'\right)$$

where g and f refer to glazed and opaque surfaces respectively and C_v is the ventilation conductance evaluated from:

$$\frac{1}{C_v} = \frac{1}{0.33 \; NV} + \frac{1}{4.8 \sum A}$$

where N is the ventilation rate (hr^{-1}), V the room volume (90 m^3) and ΣA the total internal surface area (120 m^2). From Table A8.4, N = 3 and so C_v =78.3 W/K.

From Table A8.3, the mean outside air temperature, θ_{ao}', is 16.5°C and the mean sol-air temperature, θ'_{eo}, is 23°C; therefore:

$$1126 = [2.9 \; x \; 6 + 78.3][\theta_{ei}' - 16.5] + [0.59 \; x \; 9][\theta_{ei}' - 23]$$

And therefore the mean internal environmental temperature is

$$\theta_{ei}' = 28 \; °C$$

Swing in effective solar heat gain

$$Q_s'' = S_a A_g (I_p - I') = 0.56 (3 x 2)(490 - 175) = 1058.4 W$$

where S_a is the alternating solar gain factor (Table A8.6) and I_p the peak intensity of solar radiation which here is 490 W/m^2 (i.e. the value at 14h00 allowing for a 1 hour time lag).

Structural gain

$$Q_f'' = fAU (\theta_{eo} - \theta_{eo}') = 0.3 x 5.31 (15.5 - 23) = -12 W$$

where f is the decrement factor, θ_{eo} the sol-air temperature at time of peak less time lag (i.e. at 15h00 – 8 = 07h00 = 15.5°C) and θ'_{eo} the mean sol-air temperature (= 23°C).

Casual gain

$$Q_c'' = Qc - Q_c' = 5 x 85 + 30 x 25 - 454 = 721 W$$

where Q_c is the casual gain value at the peak hour (= $g_{c1} + g_{c2}$ + ...).

Swing in gain, air-to-air

$$Q_a'' = (\sum A_g U_g + C_v)\theta_{ao}'' = (17.4 + 77.2) x 5 = 478.5 W$$

where θ_{ao} is the swing in outside air temperature (from Table A8.3 = 21.5-16.5 = 5°C).

Total swing

$$Q_t'' = 1058.4 - 12 + 721 + 478.5 = 2246 W$$

Swing in internal environmental temperature

$$Q_t'' = (\sum AY + C_v)\theta_{ei}''$$

with ΣAY = 476.19 W/K:

Element	A	Y	AY
External wall	9	0.91	8.19
Window	6	2.9	17.4
Internal walls	51	3.6	183.6
Floor	30	2.9	87
Ceiling	30	6	180
ΣAY			476.19

$$\therefore \theta_{ei}'' = 2246 / (476.19 + 78.3) = 4.1 °C$$

Peak internal environmental temperature

$$\theta_{ei} = \theta_{ei}' + \theta_{ei}'' = 28 + 4.1 = 32.1 °C$$